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Riesz Fractional Derivatives and Fractional Dimensional Space

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Abstract The Fourier transform method is used to solve fractional Poisson's equation with Riesz fractional derivative of order α. It is shown that the solution is given in terms of the fractional dimensional space D. Gauss law for the electrostatic problem is given and the total electric flux is obtained in terms of α and D.

Keywords Riesz fractional derivatives · Fractional space · Poisson's equation

1 Introduction

The history of fractional calculus [1–4], dating back to the 17th century, is almost as long as that of the integer-order calculus. Mandelbrot [5] proposed that there is a lot of fractional dimension in nature and there is a close connection between fractional Brownian motion and Riemann-Liouville fractional calculus. From then on, the fractional calculus has been used successfully to study many complex systems. The problem of electromagnetic theory was investigated by Engheta [6] and the solutions of Poisson's equations are given for the cases of fractional charge distributions in integer space. Very recently Muslih and Baleanu [7] have studied the fractional multipole expansion in fractional dimensional space using the fractional dimensional Laplacian defined by Stillinger [8].

Although the embedding space in our world is a three dimensions ($3D$) Euclidean space, the motion of material objects is not always in three dimensions. The dimensionality depends on restrain conditions [9–11].

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Fractional dimensional space has successfully been used as an effective physical description of confinement in low-dimensional system. First applied by He [9–11], this approach replaces the real confining structure with an effective space, where the measure of the anistropy or confinement is given by the non-integer dimension.

Many of the investigations into low-dimensional semiconductors structures have used a mathematical basis introduced by Stillinger [7] in which he described integration on a space of α dimensions and provided a generalization of the Laplace operator on this space. Recent progress includes the description of a single coordinate momentum operator in this fractional dimensional space based on generalized Wigner relations [12, 13] presenting realization of parastatics [14]. In some applications, the fractional dimensions appears as an explicit parameter when the physical problem is formulated in α dimensions in such a way that α maybe extended to non-integer values, as in Wilson's study of quantum field theory models in less than four dimensions [15], or in the approach to quantum mechanics by Stillinger [7]. It is worthwhile to mention that the experimental measurement of the dimensional α of our real world is given by $\alpha = (3 \pm 10^{-6})$ [7, 15]. The fractional value of α agrees with the experimental physical observations that in general relativity, gravitational fields are understood to be geometric perturbations (curvatures) in our space-time [16], rather than entities residing within a flat space-time. Besides, Zeilinger and Svozil [17] noted that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space α is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$.

The formalism from [7] has been applied to problem such as excitons [11, 18–24], magnetoexcitons [25], impurities [26], polarons [27], and superconductivity [28], often successfully mirroring computational results in specified problems. Also, the putative fractional dimension may be viewed as an effective dimension of compactified higher dimensions or as a manifestation of a non-trivial microscopic lattice structure of space [29].

In this paper we solve Poisson's equation using the Riesz fractional derivative [2, 4, 30, 31] and its Fourier transformation. Also we work out the fractional multipoles in fractional dimensional space and finding a compact expression using the definition of Gegenbauer polynomials.

2 Fractional Poisson's Equation

One of the effective methods to find the solutions of the electrostatic problems is to find the solutions of Poisson's equation and then one can find the electric field for the source charge. The starting point is to consider the fractional Poisson's equation as follows

$$(-\Delta)^{\alpha/2}\phi(\mathbf{r}) = \frac{\rho}{\epsilon_0}, \quad 1 < \alpha \leq 2, \tag{1}$$

where ρ is the source electric charge density, $\vec{r}$ is the D dimensional vector, $\Delta = \frac{\partial^2}{\partial \mathbf{r}^2}$ is the Laplacian and the operator $(-\Delta)^{\alpha/2}$ is the α dimensional generalization of fractional quantum Riesz derivative [2, 4, 30, 31]

$$(-\Delta)^{\alpha/2}\psi(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} e^{i\mathbf{k}\cdot\mathbf{r}}|\mathbf{k}|^\alpha\, d^D k \int_{R^D} \psi(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^D r. \tag{2}$$

To solve (1), we use the Fourier transform as

$$F\phi(\mathbf{r}) = g(\mathbf{k}) = \int_{R^D} \phi(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\, d^D r, \tag{3}$$

$$F\rho(\mathbf{r}) = f(\mathbf{k}) = \int_{R^D} \rho(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^D r. \tag{4}$$

The inverse Fourier transform reads as

$$F^{-1}\phi(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^D k, \tag{5}$$

$$F^{-1}\rho(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^D k. \tag{6}$$

Applying the Fourier transform with respect to spatial variable x we have

$$g(k) = \frac{f(k)}{k^\alpha}. \tag{7}$$

Hence, we obtain the solution $\phi(\mathbf{r})$ as

$$\phi(\mathbf{r}) = \int_{R^D} G(\mathbf{r}-\mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0}\, d^D r', \tag{8}$$

where $G(\mathbf{r}-\mathbf{r}')$ is the Green's function (*Kernel*) and is given by

$$G(\mathbf{r}-\mathbf{r}') = \left(\frac{1}{2\pi}\right)^D \int_{K^D} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^\alpha}\, d^D k. \tag{9}$$

Now using the following transformation defined in [2, 4]

$$\int_{K^D} \varphi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, d^D k = \int_0^\infty \varphi(\rho)\rho^{D-1}\, d\rho \int_{S_{D-1}} e^{i\rho(\mathbf{r}-\mathbf{r}')\cdot\sigma}\, d\sigma, \tag{10}$$

we obtain

$$\int_{K^D} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^\alpha}\, d^D k = \int_0^\infty \rho^{D-\alpha-1}\, d\rho \int_{S_{D-1}} e^{i\rho(\mathbf{r}-\mathbf{r}')\cdot\sigma}\, d\sigma. \tag{11}$$

Noting that

$$\int_{S_{D-1}} f(\mathbf{r}\cdot\sigma)\, d\sigma = \frac{2\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})} \int_{-1}^{1} f(|x|t)(1-t^2)^{(D-3)/2}\, dt, \tag{12}$$

and also

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{ixt}(1-t^2)^{(\nu-1/2)}\, dt, \tag{13}$$

where $J_{\nu(x)}$ is the Bessel function of the first kind. We arrive at

$$\int_{S_{D-1}} e^{ix\cdot\sigma}\, d\sigma = \frac{(2\pi)^{D/2}}{|\mathbf{x}|^{D/2-1}} J_{D/2-1}(|\mathbf{x}|). \tag{14}$$

The Green's function (9) takes the form

$$G(\mathbf{r}-\mathbf{r}') = (2\pi)^{-D/2} \int_0^\infty \frac{\rho^{D-\alpha-1} J_{D/2-1}(\rho|\mathbf{r}-\mathbf{r}'|)}{(\rho|\mathbf{r}-\mathbf{r}'|)^{D/2-1}}\, d\rho. \tag{15}$$

Using the Mellin transform of the Bessel function [2, 32]

$$\int_0^\infty \rho^\beta J_\nu(\rho)\,d\rho = \frac{2^\beta \Gamma(\frac{\nu+\beta+}{2})}{\Gamma(\frac{\beta-\nu+1}{2})}, \tag{16}$$

we obtain

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{2^\alpha \pi^{D/2}} \frac{\Gamma(\frac{D-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}}. \tag{17}$$

The solution of Poisson's equation (1) is given by

$$\phi(\mathbf{r})_{\alpha,D} = k_{\alpha,D} \int \frac{\rho(\mathbf{r}')\,d^D r'}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}}, \tag{18}$$

where the constant $k_{\alpha,D}$ is defined as

$$k_{\alpha,D} = \frac{1}{2^\alpha \pi^{D/2}\epsilon_0} \frac{\Gamma(\frac{D-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}. \tag{19}$$

For $\alpha = 2$ and $D = 3$, we have $k_{2,3} = \frac{1}{4\pi\epsilon_0}$.

For different values of α and D, we obtain the potential for any D fractional dimensional space and for any order of *Riesz* fractional derivative of order α via the relation

$$(-\Delta)^{\alpha/2}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}}\right) = \frac{2^\alpha \pi^{D/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{D-\alpha}{2})} \delta^D(\mathbf{r}-\mathbf{r}'), \tag{20}$$

where $\delta^D(\mathbf{r}-\mathbf{r}')$ is the D dimensional fractional Dirac delta function. For $\alpha = 2$ and $D = 3$ we have

$$\Delta\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \tag{21}$$

3 Gauss's Law

Gauss's law gives the relation between electric field and the charge enclosed in a closed Gauss surface. To derive Gauss's law in D dimensional fractional space, let us consider a closed D dimensional sphere of radius R with its center at the origin of the coordinate system. The total flux of the electric field on the surface of the closed sphere is

$$\oint \vec{E}\cdot d\vec{A} = \int_0^\pi \left(-\frac{\partial\phi(r)}{\partial r}\right) r^{D-1}(\sin\theta)^{D-2}\,d\theta = \frac{q r^{\alpha-2}}{2^{\alpha-1}\epsilon_0} \frac{\Gamma(\frac{D-\alpha}{2})(D-2)}{\Gamma(\frac{\alpha}{2})}. \tag{22}$$

For $\alpha = 2$ and $D = 3$

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}. \tag{23}$$

4 Fractional Multipole Expansion of Riesz Potential of Order α in Fractional D Dimensional Space

Multipole expansion of sources is a very well known subject in electromagnetism, and has been studied extensively (e.g. [33–35]). In this section we will obtain the mulipole expansion in fractional space. The potential (18) can be expanded using the definition of generating function of Gegenbauer polynomials and considering that $r > r'$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}} = \sum_{l=0}^{\infty} \frac{C_l^{(D-\alpha)/2}(\cos\vartheta) r'^l}{r^{D-\alpha+l}}, \quad r > r', \tag{24}$$

where $\hat{r} \cdot \hat{r'} = \cos\vartheta$ and $C_l^{(D-\alpha)/2}(\cos\vartheta)$ are the Gegenbauer polynomials in $\cos\vartheta$ and the forms of the first few Gegenbauer polynomials are given by

$$C_0^{(D-\alpha)/2}(x) = 1, \tag{25}$$

$$C_1^{(D-\alpha)/2}(x) = (D-\alpha)x, \tag{26}$$

$$C_2^{(D-\alpha)/2}(x) = \left(\frac{D-\alpha}{2}\right)((D-\alpha+2)x^2-1). \tag{27}$$

The potential (18), may be expressed as

$$\phi(\mathbf{r})_{\alpha,D} = k_{\alpha,D} \sum_{l=0}^{\infty} \frac{q_l^{D,\alpha}}{r^{D-\alpha+l}}, \tag{28}$$

where $q_l^{D,\alpha}$ are the fractional multipole terms of order l and are given by

$$q_l^{D,\alpha} = \int \rho(\mathbf{r}') r'^l C_l^{(D-\alpha)/2}(\cos\vartheta)\, d^D r'. \tag{29}$$

For $l = 0$ we have only fractional monopole term as

$$\phi(\mathbf{r})_{\alpha,D} = k_{\alpha,D} \frac{q}{r^{D-\alpha}}, \tag{30}$$

where q represents the monopole charge.

5 Conclusion

In this paper we have given an application of fractional calculus in electromagnetic theory and introduce a solution of the fractional Poisson's equation with Riesz derivative. The Fourier transform method is used to solve this equation and it is observed that fractional derivative and the fractional dimensional space are connected simultaneously via relation (20), which means that the fractionality in the derivatives is due to the fractionality in the space. Another interesting result we have obtained is the new definition of the constant k_α, D.

The importance of this study is that, it is considered as an intermediate cases in electromagnetic theory, where one can study the fractional multipole and the fractional space (see for example (24)). Besides it will be the starting point for given the solution of fractional Helmoltz equation $(-\Delta)^{\alpha/2}\phi(\mathbf{r},t) + (-\frac{\partial^2}{\partial t^2})^{\beta/2}\phi(\mathbf{r},t) = f(\mathbf{r},t)$, $1 < \alpha, \beta \le 2$, in D dimensional fractional space and this topic is now under consideration of the authors.

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